Solving General Equations by Order Completion

Elemér E Rosinger

Department of Mathematics
and Applied Mathematics
University of Pretoria
Pretoria
0002 South Africa
eerosinger@hotmail.com

Abstract

A method based on *order completion* for solving general equations is presented. In particular, this method can be used for solving large classes of nonlinear systems of PDEs, with possibly associated initial and/or boundary value problems.

"... provided also if need be that the notion of a solution shall be suitably extended ..."

cited from Hilbert's 20th Problem

1. Preliminaries

Recently in [3], systems of nonlinear PDEs composed of equations of the general form

$$(1.1) \quad F(x, U(x), \ldots, D_x^p U(x), \ldots) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^n$$

were solved on domains Ω that can be any open, not necessarily

bounded subsets of $\mathbb{R}^n$, while $p \in \mathbb{N}^n,\ |p| \leq m$, with the orders $m \in \mathbb{N}$ of the PDEs arbitrary given.
The *unprecedented generality* of these nonlinear systems of PDEs comes, above all, from the class of functions F which define the left hand terms, and which are only assumed to be *jointly continuous* in all of their arguments. The right hand terms f are also required to be *continuous* only.
However, with minimal modifications of the method, both F and f can have certain *discontinuities* as well, [3].
Regardless of the above generality of the nonlinear systems of PDEs considered, and of possibly associated initial and/or boundary value problems, one can always find for them solutions U defined on the *whole* of the respective domains Ω. These solutions U have the *blanket, type independent*, or *universal regularity* property that they can be assimilated with *Hausdorff continuous functions*, [1,4-6].
It follows in this way that, when solving systems of nonlinear PDEs of the generality of those in (1.1), one can *dispense with* the various customary spaces of distributions, hyperfunctions, generalized functions, Sobolev spaces, and so on. Instead one can stay within the realms of *usual functions*. Also, when proving the *existence* and the mentioned type of *regularity* of such solutions one can dispense with methods of Functional Analysis. However, functional analytic methods can possibly be used in order to obtain further regularity or other desirable properties of such solutions.

The mentioned generality of the equations solved and the regularity of the solutions obtained is based on the use of the *order completion method*, first introduced and developed in [3].
As it happens, however, this order completion method reaches far beyond the solution of systems of nonlinear PDEs, and in fact it can be applied to the solution of the surprisingly general equations

$$(1.2) \quad T(A) \;=\; F$$

where

$$(1.3) \quad T : X \;\longrightarrow\; Y$$

is any mapping, X is any nonvoid set, while $(Y, \leq)$ is a partially ordered set, or in short, poset, while $F \in Y$ is given, and $A \in X$ is the sought after solution.

Needless to say, in general, for a given $F \in Y$, there may not exist any solution $A \in X$ for the equation (1.2). Consequently, the setup (1.2), (1.3) may have to be *extended*.
Customarily, such extensions assume suitable topologies on X and Y, and certain continuity properties for the mapping T in (1.3).
However, as shown in [3], and also seen in the sequel, for the same purpose of solving the equations (1.2) in an extended setup, one can successfully use the *order completion* of the spaces X and Y.

And one of the major advantages of such an approach is that such a method

- does *no longer* differentiate between linear and nonlinear operators T in (1.3), in case the spaces X and Y may happen to have a linear vector space structure, [3].

Two particularly convenient further features of the order completion method in solving general equations (1.2) are the following :

- one obtains necessary and sufficient conditions for the existence of solutions,

- one obtains explicit expressions of the solutions, whenever they exist.

We shall present one of the general approaches resulting from the order completion method for solving equations of type (1.2). Further possible developments in this regard of the order completion method will be indicated.

2. Pull-Back Order

Without loss of generality, [3], we shall assume that all the posets considered are without a minimum or maximum element. Various notions

and results related to partial orders which are used in the sequel are presented in the Appendix.

Given an equation (1.2), (1.3), we define on X the equivalence relation $\approx_T$ by

$$(2.1) \quad u \approx_T y \iff T(u) = T(v)$$

for $u, v \in X$. In this way, by considering the quotient space

$$(2.2) \quad X_T = X / \approx_T$$

we obtain the *injective* mapping

$$(2.3) \quad T_\approx : X_T \longrightarrow Y$$

defined by

$$(2.4) \quad X_T \ni U \longmapsto T(u) \in Y$$

where $u \in U$, that is, U is the $\approx_T$ equivalence class of u in X_T, while $T(u)$ is defined by (1.3).

At that stage, we can define a partial order $\leq_T$ on X_T as being the *pull-back* by the mapping $T_\approx$ in (2.3) of the given partial order $\leq$ on Y, namely

$$(2.5) \quad U \leq_T V \iff T_\approx(U) \leq T_\approx(V)$$

for $U, V \in X_T$. The effect of the above construction is that we obtain the *order isomorphic embedding*, or in short OIE

$$(2.6) \quad T_\approx : X_T \longrightarrow Y$$

As mentioned, without loss of generality we shall assume that the poset $(X_T^\#, \leq_T)$ has no minimum or maximum.

And now, we consider the *order completions* $X_T^\#$ and $Y^\#$ of X_T, and

respectively, Y.
For simplicity, we shall denote by $\leq$ the partial orders both on $X_T^\#$ and $Y^\#$. In fact, as seen in (A.20), these partial orders are the usual inclusion relations $\subseteq$ among subsets of X, respectively, of Y.

Then according to Proposition A.1 in the Appendix, we obtain the commutative diagram of OIE-s

$$
\begin{tikzcd}
X_T \arrow[r, "T_\approx"] \arrow[d, "\subseteq"] & Y \arrow[d, "\subseteq"'] \\
X_T^\# \arrow[r, "T^\#"'] & Y^\#
\end{tikzcd}
\tag{2.7}
$$

Consequently, for $U \in X_T$ and $A \in X_T^\#$, we have in $Y^\#$ the relations

$$(2.8) \quad T^\#(< U\,]\,) \;=\; < T_\approx(U)\,]$$

$$(2.9) \quad T^\#(A) \;=\; (T_\approx(A))^{ul} \;=\; \sup{}_{Y^\#} \;\{\; < T_\approx(U)\,] \;\mid\; U \in A \;\}$$

3. Reformulation

Now we can reformulate the problem of solving the general equations (1.2) as follows. Given $F \in Y^\#$, find *necessary and sufficient* conditions for the existence of $A \in X_T^\#$, such that

$$(3.1) \quad T^\#(A) \;=\; F$$

4. Solution

We note that (2.7) gives the inclusions

$$
(4.1) \quad \begin{aligned}
& \sup_{Y^\#} \; \{ \; T^\#(U) \mid U \in X^\#_T, \; T^\#(U) \subseteq F \; \} \; \subseteq \\
& \subseteq \; T^\#(\; \sup_{X^\#_T} \; \{ \; U \mid U \in X^\#_T, \; T^\#(U) \subseteq F \; \} \;) \; \subseteq \\
& \subseteq \; T^\#(\; \inf_{X^\#_T} \; \{ \; V \mid V \in X^\#_T, \; F \; \subseteq \; T^\#(V) \; \} \;) \; \subseteq \\
& \subseteq \; \inf_{Y^\#} \; \{ \; T^\#(V) \mid V \in X^\#_T, \; F \; \subseteq \; T^\#(V) \; \}
\end{aligned}
$$

Indeed, the first and last inclusions follow from Lemma A.1 in the Appendix. As for the middle inclusion in (4.1), let $U, V \in X^\#_T$ be such that $T^\#(U) \subseteq F \subseteq T^\#(V)$. Then $T^\#(U) \subseteq T^\#(V)$, hence $U \leq V$, since $T^\#$ is an OIE. It follows that

$$
\begin{aligned}
& \sup_{X^\#_T} \; \{ \; U \mid U \in X^\#_T, \; T^\#(U) \subseteq F \; \} \; \leq \\
& \quad \leq \; \inf_{X^\#_T} \; \{ \; V \mid V \in X^\#_T, \; F \; \subseteq \; T^\#(V) \; \}
\end{aligned}
$$

and the proof of (4.1) is completed.

We note further that the above inequality $T^\#(U) \subseteq F \subseteq T^\#(V)$ also implies

$$
(4.2) \quad \begin{aligned}
& \sup_{Y^\#} \; \{ \; T^\#(U) \mid U \in X^\#_T, \; T^\#(U) \subseteq F \; \} \; \subseteq \; F \; \subseteq \\
& \qquad\qquad \subseteq \; \inf_{Y^\#} \; \{ \; T^\#(V) \mid V \in X^\#_T, \; F \; \subseteq \; T^\#(V) \; \}
\end{aligned}
$$

Furthermore

$$
(4.3) \quad \{ \; U \in X^\#_T \mid T^\#(U) \; \subseteq \; F \; \} \; \neq \; \phi
$$

since (A.9) gives $U = \phi \in X^\#_T$, hence in view of (A.4) - (A.6) and (2.9) we have $T^\#(U) = (T(\phi))^{ul} = \phi^{ul} = (\phi^u)^l = (Y^\#)^l = \phi \subseteq F$.

Returning now to the problem (3.1), we note that it is not trivial. Indeed, the OIE in (2.7), namely

$$
T^\# : X^\#_T \; \longrightarrow \; Y^\#
$$

need *not* be surjective. Further $T^{\#}$ need *not* preserve infima or suprema.

However, in the next theorem we can obtain the following two general results :

- a necessary and sufficient condition for the solvability of (9,12), and
- the explicit expression of the solution, when it exists.

Theorem 4.1.

Given $F \in Y^{\#}$.

1) The equation

$$(4.4) \quad T^{\#}(A) \;=\; F$$

has a solution $A \in X^{\#}_T$, if and only if, see (4.1)

$$(4.5) \quad \begin{aligned} &\sup\nolimits_{Y^{\#}} \; \{\, T^{\#}(U) \mid U \in X^{\#}_T,\; T^{\#}(U) \subseteq F \,\} \;= \\ &\qquad\qquad = \; \inf\nolimits_{Y^{\#}} \; \{\, T^{\#}(V) \mid V \in X^{\#}_T,\; F \;\subseteq\; T^{\#}(V) \,\} \end{aligned}$$

2) This solution is unique, whenever it exists, see (2.7).

3) When it exists, the unique solution $A \in X^{\#}_T$ is given by

$$(4.6) \quad \begin{aligned} A \;&=\; \sup\nolimits_{X^{\#}_T} \; \{\, U \in X^{\#}_T \mid T^{\#}(U) \;\subseteq\; F \,\} \;= \\ &=\; \inf\nolimits_{X^{\#}_T} \; \{\, V \in X^{\#}_T \mid F \;\subseteq\; T^{\#}(V) \,\} \end{aligned}$$

and, see (4.3)

$$(4.7) \quad \{\, U \in X^{\#}_T \mid T^{\#}(U) \;\subseteq\; F \,\},\;\; \{\, V \in X^{\#}_T \mid F \;\subseteq\; T^{\#}(V) \,\} \;\neq\; \phi$$

Proof

From (4.4) follows that $A \in X^{\#}_T,\; T^{\#}(A) \subseteq F$, thus

$$F \;=\; T^{\#}(A) \;\subseteq\; \sup_{Y^{\#}} \;\{\; T^{\#}(U) \mid U \in X_T^{\#},\; T^{\#}(U) \subseteq F \;\}$$

Similarly we have

$$\inf_{Y^{\#}} \;\{\; T^{\#}(V) \mid V \in X_T^{\#},\; F \;\subseteq\; T^{\#}(V) \;\} \;\subseteq\; T^{\#}(A) \;=\; F$$

thus (4.1) collapses to the seven equalities

$$F \;=\; T^{\#}(A) \;=\; \sup_{Y^{\#}} \;\ldots\; \;=\; T^{\#}(\sup_{X_T^{\#}} \;\ldots\;) \;=$$

$$=\; T^{\#}(\inf_{X_T^{\#}} \;\ldots\;) \;=\; \inf_{Y^{\#}} \;\ldots\; \;=\; T^{\#}(A) \;=\; F$$

Thus in particular we obtain (4.5).

The injectivity of $T^{\#}$ will give (4.6), while (4.7) follows from (4.3) and the fact that we can take $V = A$.

Conversely, let us assume (4.5). Then (4.1) collapses to the three equalities

$$\sup_{Y^{\#}} \;\ldots\; \;=\; T^{\#}(\sup_{X_T^{\#}} \;\ldots\;) \;=\; T^{\#}(\inf_{X_T^{\#}} \;\ldots\;) \;=\; \inf_{Y^{\#}} \;\ldots$$

thus in view of the corresponding collapsed version of (4.2), we can extend the above three equalities to the following four

$$\sup_{Y^{\#}} \;\ldots\; \;=\; T^{\#}(\sup_{X_T^{\#}} \;\ldots\;) \;=\; T^{\#}(\inf_{X_T^{\#}} \;\ldots\;) \;=\; \inf_{Y^{\#}} \;\ldots\; \;=\; F$$

And now the injectivity of $T^{\#}$ will give (4.4) and (4.6), while (4.7) follows as above.

$\square$

The above existence result is of a "local" nature, since it refers to a solution of the equation (3.1) for one given right hand term $F \in Y^{\#}$. This result, however, can further be strengthened by the following "global" one which characterizes the solvability of (3.1) for *all* right hand terms $F \in Y^{\#}$. Namely, we have, [3, pp. 190,191]

Theorem 4.2.

The following are equivalent

(4.8) $\quad T^\#(X_T^\#) \supseteq Y$

and

(4.9) $\quad T^\#(X_T^\#) = Y^\#$

In each of these cases $T^\#$ is an order isomorphism, or in short, OI, between $X_T^\#$ and $Y^\#$.

It is *important* to note the following two fact :

- "Pull-back" type structures are customary when solving PDEs by functional analytic methods. Details in this regard are presented in [3, chap. 12], while one well known classical example can be seen in section 7 in the sequel.

- As shown in [3, chap. 13], one can consider in (2.7) far more general partial orders than the "pull-back" type ones, and still obtain solutions for nonlinear PDEs in (1.1) by the order completion method.

5. Applications to Nonlinear Systems of PDEs

Let us now associate with a nonlinear PDE in (1.1) the corresponding nonlinear partial differential operator defined by the left hand side, namely

(5.1) $\quad T(x, D)U(x) = F(x, U(x), \ldots, D_x^p U(x), \ldots), \quad x \in \Omega$

Two facts about the nonlinear PDEs in (1.1) and the corresponding nonlinear partial differential operators $T(x, D)$ in (5.1) are important and immediate :

- The operators $T(x, D)$ can *naturally* be seen as acting in the *classical* context, namely

$$(5.2) \qquad T(x, D) \; : \; \mathcal{C}^m(\Omega) \ni U \quad \longmapsto \quad T(x, D)U \in \mathcal{C}^0(\Omega)$$

while, unfortunately on the other hand :

- The mappings in this natural classical context (5.2) are typically *not* surjective even in the case of linear $T(x, D)$, and they are even less so in the general nonlinear case of (1.1).

In other words, linear or nonlinear PDEs in (1.1) typically *cannot* be expected to have *classical* solutions $U \in \mathcal{C}^m(\Omega)$, for arbitrary continuous right hand terms $f \in \mathcal{C}^0(\Omega)$, as illustrated by a variety of well known examples, some of them rather simple ones, see [3, chap. 6].
Furthermore, it can often happen that nonclassical solutions do have a major applicative interest, thus they have to be sought out *beyond* the confines of the classical framework in (5.2).
This is, therefore, how we are led to the *necessity* to consider *generalized solutions* U for PDEs like those in (1.1), that is, solutions $U \notin \mathcal{C}^m(\Omega)$, which therefore are no longer classical. This means that the natural classical mappings (5.2) must in certain suitable ways be *extended* to *commutative diagrams*

$$(5.3) \qquad \begin{array}{ccc} \mathcal{C}^m(\Omega) & \xrightarrow{\quad T(x, D) \quad} & \mathcal{C}^0(\Omega) \\ \big\downarrow \subseteq & & \subseteq \big\downarrow \\ X & \xrightarrow[\quad \overline{T} \quad]{} & Y \end{array}$$

with the generalized solutions now being found as

$$(5.4) \qquad U \in X \setminus \mathcal{C}^m(\Omega)$$

instead of the classical ones $U \in \mathcal{C}^m(\Omega)$ which may easily fail to exist.
A further important point is that one expects to reestablish certain kind of *surjectivity* type properties typically missing in (5.2), at least

such as for instance

$$(5.5) \quad \mathcal{C}^0(\Omega) \subseteq \overline{T}(X)$$

As it turns out, when constructing extensions of (5.2) given by commutative diagrams (5.3), we shall be interested in the following somewhat larger spaces of piecewise smooth functions. For any integer $0 \leq l \leq \infty$, we define

$$(5.6) \quad \mathcal{C}^l_{nd}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \exists\, \Gamma \subset \Omega \text{ closed, nowhere dense } : \\ \quad u \in \mathcal{C}^l(\Omega \setminus \Gamma) \end{array} \right\}$$

and as an immediate strengthening of (5.2), we obviously obtain

$$(5.7) \quad T(x,D)\, \mathcal{C}^m_{nd}(\Omega) \subseteq \mathcal{C}^0_{nd}(\Omega)$$

The solution of the nonlinear PDEs in (1.1) through the order completion method will come from the construction of specific instances of the *commutative diagrams* (5.3), given by

$$(5.8) \quad \begin{array}{ccc} \mathcal{C}^m_{nd}(\Omega) & \xrightarrow{\quad T(x,D) \quad} & \mathcal{C}^0_{nd}(\Omega) \\ \downarrow & & \downarrow \\ \mathcal{M}^m_T(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{M}^0(\Omega) \\ \downarrow & & \downarrow \\ \mathcal{M}^m_T(\Omega)^\# & \xrightarrow[\quad T^\# \quad]{\text{bijective}} & \mathcal{M}^0(\Omega)^\# \end{array}$$

where the operation $(\quad)^\#$ means the *order completion*, [3], of the respective spaces, as well as the extension to such order completions of the respective mappings, see (2.7). It follows that in terms of (5.3), we have

$$X = \mathcal{M}^m_T(\Omega)^\#, \quad Y = \mathcal{M}^0(\Omega)^\#, \quad \overline{T} = T^\#$$

thus we shall obtain for the nonlinear PDEs in (1.1) generalized solutions

$$(5.9) \quad U \in \mathcal{M}_T^m(\Omega)^\#$$

Furthermore, instead of the *surjectivity* condition (5.5), we shall at least have the following stronger one

$$(5.10) \quad \mathcal{C}^0_{nd}(\Omega) \subseteq T^\#(\mathcal{M}_T^m(\Omega)^\#)$$

So far about the main ideas related to the *existence* of solutions of general nonlinear PDEs of the form (1.1). Further details can be found in [3,1,4-6].

As for the *regularity* of such solutions, we recall that, as shown in [1], one has the inclusions

$$(5.11) \quad \mathcal{M}^0(\Omega)^\# \subseteq Mes\,(\Omega)$$

where $Mes\,(\Omega)$ denotes the set of Lebesgue measurable functions on Ω. In this way, in view of (5.8) and (5.9), one can assimilate the generalized solutions U of the nonlinear PDEs in (1.1) with usual measurable functions in $Mes\,(\Omega)$.
Recently, however, based on results in [1,4-6], it was shown that instead of (5.11), one has the much *stronger regularity* property

$$(5.12) \quad \mathcal{M}^0(\Omega)^\# \subseteq \mathbb{H}\,(\Omega)$$

where $\mathbb{H}\,(\Omega)$ denotes the set of Hausdorff continuous functions on Ω. Consequently, now one can significantly improve on the earlier regularity result, as one can assimilate the generalized solutions U of the nonlinear PDEs in (1.1) with usual functions in $\mathbb{H}\,(\Omega)$.

Regarding *systems* of nonlinear PDEs such as in (1.1), with possibly associated initial and/or boundary value problems, it was shown in [3, chap. 8] the way they can be dealt with the above order completion method.

In this respect, a *surprising* advantage of the order completion method is the ease, when compared with the usual functional analytic approaches, in dealing with initial and/or boundary value problems.

6. Beyond "Pull-Back" Partial Orders

As mentioned at the end of section 4, and presented in full detail in [3, chap. 13], the order completion method in solving large classes of nonlinear systems of PDEs of the type (1.1) is *not* limited to the use of "pull-back" type partial orders in (2.7), (5.3) and (5.8). In fact, a large class of more general partial orders can be defined on the domains $\mathcal{M}_T^m(\Omega)$ of the respective PDEs, and stil obtain for them solutions in the corresponding order completions.

7. Use of "Pull-Back" in Functional Analytic Solution Methods

As presented in detail in [3, chap. 12], functional analytic methods used for solving PDEs do often employ topologies obtained by "pull-back". Here we present shortly one of the classical such examples. Let us consider on a bounded Euclidean domain Ω, which has a smooth boundary $\partial\,\Omega$, the following familiar linear boundary value problem, usually called the Poisson Problem

$$(7.1)\quad \begin{aligned} &\Delta\, U(x) \;=\; f(x), \quad x \in \Omega \\ &U \;=\; 0 \quad \text{on} \;\; \partial\,\Omega \end{aligned}$$

As is well known, for every given $f \in C^\infty(\overline{\Omega})$, where $\overline{\Omega}$ denotes the closure of Ω, this problem has a unique solution U in the space

$$(7.2)\quad X \;=\; \left\{ \; v \in C^\infty(\overline{\Omega}) \;\mid\; v \;=\; 0 \;\; \text{on} \;\; \partial\,\Omega \; \right\}$$

It follows that the mapping

$$(7.3)\quad X \ni v \;\longmapsto\; ||\; \Delta v \;||_{\,L^2(\,\Omega\,)}$$

defines a norm on the vector space X. Now let

$$(7.4) \quad Y \;=\; C^\infty(\overline{\Omega})$$

be endowed with the topology induced by $L^2(\Omega)$. Then in view of (7.1) - (7.4), the mapping

$$(7.5) \quad \Delta : X \;\rightarrow\; Y$$

is a uniform continuous linear bijection. Therefore, it can be extended in a unique manner to an isomorphism of Banach spaces

$$(7.6) \quad \Delta : \overline{X} \;\rightarrow\; \overline{Y} \;=\; L^2(\Omega)$$

In this way one has the classical existence and uniqueness result

$$\begin{array}{ll} & \forall \quad f \in L^2(\Omega) \quad : \\ (7.7) & \quad \exists\,! \;\; U \in \overline{X} \quad : \\ & \qquad\quad \Delta U = f \end{array}$$

The power and simplicity - based on linearity and topological completion of uniform spaces - of the above classical existence and uniqueness result is obvious. This power is illustrated by the fact that the set $\overline{Y} = L^2(\Omega)$ in which the right hand terms f in (7.1) can now be chosen is much *larger* than the original $Y = C^\infty(\overline{\Omega})$. Furthermore, the existence and uniqueness result in (7.7) does not need the a priori knowledge of the structure of the elements $U \in \overline{X}$, that is, of the respective generalized solutions. This structure which gives the regularity properties of such solutions can be obtained by a further detailed study of the respective differential operators defining the PDEs under consideration, in this case, the Laplacean Δ. And in the above specific instance we obtain

$$(7.8) \quad \overline{X} \;=\; H^2(\Omega) \cap H^1_0(\Omega)$$

As seen above, typically for the functional analytic methods, the generalized solutions are obtained in topological completions of vector

spaces of usual functions. And such completions, like for instance the various Sobolev spaces, are defined by certain linear partial differential operators which may happen to *depend* on the PDEs under consideration.

In the above example, for instance, the topology on the space X obviously *depends* on the specific PDE in (7.1). Thus the topological completion $\overline{X}$ in which the generalized solutions U are found according to (7.7), does again *depend* on the respective PDE.

Appendix

We shortly present several notions and results used above. A related full presentation can be found in [3, Appendix, pp. 391-420].

Let $(X, \leq)$ be a nonvoid poset without minimum or maximum. For $a \in X$ we denote

(A.1) $\quad < a] = \{x \in X \mid x \leq a\}, \quad [a >= \{x \in X \mid x \geq a\}$

We define the mappings

(A.2) $\quad X \supseteq A \longmapsto A^u = \bigcap_{a \in A} [a > \ \subseteq X$

(A.3) $\quad X \supseteq A \longmapsto A^l = \bigcap_{a \in A} < a] \subseteq X$

then for $A \subseteq X$ we have

(A.4) $\quad A^u = X \Longleftrightarrow A^l = X \Longleftrightarrow A = \phi$

(A.5) $\quad A^u = \phi \Longleftrightarrow A$ unbounded from above

(A.6) $\quad A^l = \phi \Longleftrightarrow A$ unbounded from below

Definition A.1.

We call $A \subseteq X$ a *cut*, if and only if

(A.7) $\quad A^{ul} = A$

and denote

(A.8) $\quad X^{\#} = \{A \subseteq X \mid A \text{ is a cut}\} \subseteq \mathcal{P}(X)$

$\square$

Clearly, (A.4) - (A.6) imply

(A.9) $\quad \phi,\ X \in X^{\#}$

therefore

(A.10) $\quad X^{\#} \neq \phi$

Given $A, B \subseteq X$, we have

(A.11) $\quad A \subseteq B \Longrightarrow A^u \supseteq B^u,\ A^l \supseteq B^l$

(A.12) $\quad A \subseteq A^{ul}, \quad A \subseteq A^{lu}$

(A.13) $\quad A^{ulu} = A^u, \quad A^{lul} = A^l$

Consequently

$$\forall \ A \subseteq X \ :$$

$$*) \ A^{ul} \in X^{\#}$$

(A.14) $$**) \ \forall \ B \in X^{\#} \ :$$

$$A \subseteq B \Longrightarrow A^{ul} \subseteq B$$

$$B \subseteq A \Longrightarrow B \subseteq A^{ul}$$

therefore

(A.15) $\quad X^{\#} = \{A^{ul} \mid A \subseteq X\}$

Given $x \in X$, we have

(A.16) $\quad \{x\}^u = [x >, \quad \{x\}^l =< x], \quad [x >^l =< x], \quad < x]^u = [x >$

(A.17) $\quad \{x\}^{ul} =< x], \quad \{x\}^{lu} = [x >$

We denote for short

$\{x\}^u = x^u, \;\; \{x\}^l = x^l, \;\; \{x\}^{ul} = x^{ul}, \;\; \{x\}^{lu} = x^{lu}, \; \ldots$

Given $A \in X^{\#}$, we have

(A.18) $$\phi \neq A \neq X \Longleftrightarrow \left(\begin{array}{l} \exists \;\; a, b \in X \;\; : \\ \\ \quad < a] \;\subseteq\; A \;\subseteq\; < b] \end{array} \right)$$

We shall use the *embedding*

(A.19) $\quad X \ni x \stackrel{\varphi}{\longmapsto} x^{ul} = x^l =< x] \in X^{\#}$

We define on $X^{\#}$ the partial order

(A.20) $\quad A \leq B \Longleftrightarrow A \subseteq B$

Definition 2.1.

Given two posets $(X, \leq)$, $(Y, \leq)$ and a mapping $\varphi : X \longrightarrow Y$. We call φ an *order isomorphic embedding*, or in short, OIE, if and only if it is injective, and furthermore, for $a, b \in X$ we have

$$a \;\leq\; b \quad \Longleftrightarrow \quad \varphi(a) \;\leq\; \varphi(b)$$

An OIE φ is an *order isomorphism*, or in short, OI, if and only if it is bijective.

$\square$

The main result concerning order completion is given in, [2] :

Theorem (H M MacNeille, 1937)

1) The poset $(X^\#, \leq)$ is order complete.

2) The embedding $X \xrightarrow{\varphi} X^\#$ in (A.19) preserves infima and suprema, and it is an order isomorphic embedding, or OIE.

3) For $A \in X^\#$, we have the order density property of X in $X^\#$, namely

$$\text{(A.21)} \quad \begin{aligned} A \;&=\; \sup_{X^\#} \; \{ x^l \mid x \in X, \;\; x^l \subseteq A \} \;= \\ &=\; \inf_{X^\#} \; \{ x^l \mid x \in X, \;\; A \subseteq x^l \} \end{aligned}$$

$\square$

For $A \subseteq X$, we have

$$\text{(A.22)} \quad A^{ul} = \sup_{X^\#} \; \{ x^l \mid x \in A \}$$

Given $A_i \in X^\#$, with $i \in I$, we have with the partial order in $X^\#$ the relations

$$\text{(A.23)} \quad \sup_{i \in I} \; A_i \;=\; \inf \; \{ A \in X^\# \mid \bigcup_{i \in I} A_i \subseteq A \} \;=\; (\bigcup_{i \in I} A_i)^{ul}$$

$$\text{(A.24)} \quad \begin{aligned} \inf_{i \in I} \; A_i \;&=\; \sup \; \{ A \in X^\# \mid A \subseteq \bigcap_{i \in I} A_i \} \;=\; (\bigcap_{i \in I} A_i)^{ul} \;= \\ &=\; \bigcap_{i \in I} A_i \end{aligned}$$

Extending mappings to order completions

Let $(X, \leq), \; (Y, \leq)$ be two posets without minimum or maximum, and let

$$\text{(A.25)} \quad \varphi : X \longrightarrow Y$$

be any mapping. Our interest is to obtain an extension

$$\varphi^{\#} : X^{\#} \longrightarrow Y^{\#}$$

For that, we first extend φ to a *larger* domain, as follows

$$(\text{A.26}) \quad \varphi^{\#} : \mathcal{P}(X) \longrightarrow Y^{\#}$$

where for $A \subseteq X$ we define

$$(\text{A.27}) \quad \varphi^{\#}(A) \;=\; (\varphi(A))^{ul} \;=\; \sup_{Y^{\#}} \{< \varphi(x)\,] \mid x \in A\}$$

and for any mapping in (A.25), we obtain the commutative diagram

$$(A.28) \quad \begin{tikzcd} X \ni x \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto] & \varphi(x) \in Y \arrow[d, mapsto] \\ \mathcal{P}(X) \ni \{x\} \arrow[r, mapsto, "\varphi^{\#}"'] & \varphi^{\#}(x) \;=\; < \varphi(x)\,] \in Y^{\#} \end{tikzcd}$$

Proposition A.1.

1) The mapping $\varphi^{\#} : \mathcal{P}(X) \longrightarrow Y^{\#}$ in (A.36) is increasing, if on $\mathcal{P}(X)$ we take the partial order defined by the usual inclusion "$\subseteq$".

2) If the mapping $\varphi : X \longrightarrow Y$ in (A.35) is increasing, then the mapping $\varphi^{\#} : \mathcal{P}(X) \longrightarrow Y^{\#}$ in (A.36) is an extension of it to $X^{\#}$, namely, we have the commutative diagram

$$(A.29) \quad \begin{tikzcd} X \ni x \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto] & \varphi(x) \in Y \arrow[d, mapsto] \\ X^{\#} \ni < x] \arrow[r, mapsto, "\varphi^{\#}"'] & \varphi^{\#}(< x\,]\,) \;=\; < \varphi(x)\,] \in Y^{\#} \end{tikzcd}$$

3) If the mapping $\varphi : X \longrightarrow Y$ in (A.25) is an OIE, then the mapping $\varphi^\# : \mathcal{P}(X) \longrightarrow Y^\#$ in (A.26) when restricted to $X^\#$, that is

(A.30) $\quad \varphi^\# : X^\# \longrightarrow Y^\#$

as in (A.29), is also an OIE.

Lemma A.1.

Let in general $\mu : M \longrightarrow N$ be an increasing mapping between two order complete posets, then for nonvoid $E \subseteq M$ we have

(A.31) $\quad \mu(\inf_M E) \;\leq\; \inf_N \mu(E) \;\leq\; \sup_N \mu(E) \;\leq\; \mu(\sup_M E)$

Proof

Indeed, let $a = \inf_M E \in M$. Then $a \leq b$, with $b \in E$. Hence $\mu(a) \leq \mu(b)$, with $b \in E$. Thus $\mu(a) \leq \inf_N \mu(E)$, and the first inequality is proved.
The last inequality is obtained in a similar manner, while the middle inequality is trivial.

$\square$

References

[1] Anguelov R, Rosinger E E : Hausdorff continuous solutions of nonlinear PDEs through the order completion method. Quaestiones Mathematicae, Vol. 28, 2005, 1-15, arXiv : math.AP/0406517

[2] Luxemburg W A J, Zaanen A C : Riesz Spaces, I. North-Holland, Amsterdam, 1971

[3] Oberguggenberger M B, Rosinger E E : Solution of Continuous Nonlinear PDEs through Order Completion. Mathematics Studies VOl. 181, North-Holland, Amsterdam, 1994

[4] Rosinger E E : Hausdorff continuous solutions of arbitrary continuous nonlinear PDEs through the order completion method. math.AP/0405546

[5] Rosinger E E : Can there be a general nonlinear PDE theory for the existence of solutions ? math.AP/0407026

[6] Rosinger E E : Solving large classes of nonlinear systems of PDEs. math.AP/0505674